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THE ONE-DIMENSIONAL POROUS MEDIUM EQUATION WITH CONVECTION: CONTINUOUS DIFFERENTIABILITY OF INTERFACES AFTER THE WAITING TIME

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Abstract—This paper deals with the Cauchy problem for degenerate parabolic equation $u_t = (u^m)_{xx} - (u^n)_x$ with $n \geq m > 1$. We show the C^1-regularity of the interfaces after the waiting time.

1. INTRODUCTION

In this note we consider the Cauchy problem

$$(P)\quad \begin{cases} u_t = (u^m)_{xx} - (u^n)_x, & (x,t) \in Q := \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with $n \geq m > 1$ and where u_0 is a nonnegative continuous function with compact support $[\zeta_{01}, \zeta_{02}]$.

Equation (1.1) models a number of different physical phenomena. For instance, when u denotes an unsaturated soil-moisture content, the equation describes the infiltration of water in a homogeneous porous medium.

Problem (P) has been studied by several authors. In particular we refer to Gilding [1,2] and to the references in those papers. We define $P[u] := \{(x,t) \in Q : u(x,t) > 0\}$. Then there exist two functions $\zeta_i(t) \in C^{1/2}([0,\infty]) \cap C^{0+1}((0,\infty))$, $i = 1,2$ such that $\zeta_i(0) = \zeta_{0i}$ and $P_t := \{x \in \mathbb{R}, (x,t) \in P[u]\}$ is the interval $(\zeta_1(t), \zeta_2(t))$. Furthermore $(-1)^i\zeta_i(t)$ is nondecreasing and $\lim_{t\to\infty} (-1)^i\zeta_i(t) = +\infty$ for $i = 1,2$.

The purpose of this note is to prove the following result:

THEOREM 1. *Suppose that $n \geq m > 1$ and that u_0 is a nonnegative continuous function with compact support $[\zeta_{01}, \zeta_{02}]$. Then the waiting time $t_i^* = Sup\ \{t > 0,\ \zeta_i(t) = \zeta_{0i}\}$, $i = 1,2$, which is finite and possibly zero, is such that*

(i) *$(-1)^i\zeta_i(t)$ is strictly increasing for $t > t_i^*$,*
(ii) *$\zeta_i \in C^1((t_i^*, \infty))$.*

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The idea of the method is to prove first that

$$(-1)^i \left(\zeta_i'' + \frac{C}{t}\zeta_i'\right) \geq 0, \qquad i = 1,2 \qquad \text{in the sense of distributions,}$$

where C is a positive constant, as is done by Bertsch, van Duijn, Esteban and Zhang Hongfei [3] in a slightly different context.

Note that properties (i) and (ii) have been proved by Herrero and Vazquez [4] for a nonlinear diffusion equation with absorption. Finally we refer to Schmariev [5] and Diaz and Schmariev [6] for the proof of the Lipschitz continuity of the interface for Problem (P) in the case that $0 < n < 1$.

2. A DIFFERENTIAL INEQUALITY FOR THE FREE BOUNDARY

We start with some estimates on the solution u of Problem (P).

LEMMA 1. *For a.e. $(x,t) \in Q$ there holds*

$$\left|(u^{m-1})_x(x,t)\right|^2 \leq C \left\|u_0\right\|_{L^\infty(\mathbb{R})}^{m-1} t^{-1}.$$

In order to prove Lemma 1, one uses a Bernstein argument as is done for instance by Herrero and Vazquez [4].

Next we set

$$V(x,t) = -\frac{1}{u(x,t)} \left\{(u^m(x,t))_x - u^n(x,t)\right\}, \tag{2.1}$$

for all $t \in (0,\infty)$ and $x \in (\zeta_1(t), \zeta_2(t))$. In particular we derive from (1.1) that u is a weak solution of

$$u_t + (uV)_x = 0 \qquad \text{in } Q. \tag{2.2}$$

Note that by Lemma 1, V is bounded in $\bar{\mathbb{R}} \times [\delta, \infty)$ for all $\delta > 0$.

LEMMA 2. *For every $T > 0$ there exists a positive constant $C = C(T)$ such that*

$$V_x \leq \frac{C}{t} \text{ in the sense of distributions in } Q_T := \mathbb{R} \times (0,T). \tag{2.3}$$

PROOF. The function $q = -V_x$ satisfies a differential equation of the form $\mathcal{L}q = 0$ where

$$\mathcal{L}q = Aq_{xx} + Bq_x + (m+1)q^2 + \frac{(m-n+1)(n-1)}{m-1} u^{n-m}(u^{m-1})_x q - q_t.$$

We set $\underline{q} = -C/t$ and deduce from Lemma 1 that $\mathcal{L}\underline{q} \geq 0$ on Q_T provided that $C \geq (1 + K\sqrt{T})/(m-1)$. Thus $q \geq \underline{q}$ on Q_T.

COROLLARY 1. *Let $D^+\zeta_i(t)$, $i = 1,2$ denote the right derivative of ζ_i at t.*

$$D^+\zeta_i(t) = \lim_{h \downarrow 0} \frac{\zeta_i(t+h) - \zeta_i(t)}{h} = \lim_{\substack{x \to \zeta_i(t) \\ (x,t) \in P[u]}} \{V(x,t)\}, \qquad \text{for all } t > 0. \tag{2.4}$$

PROOF. We deduce from Lemma 2 that the limit on the right-hand side of (2.4) exists. The result then follows from Gilding [2, Theorem 4].

Next we define Lagrangian coordinates. Suppose that $\int_{\mathbb{R}} u_0(x)\,dx = 1$. Let $y \in [0,1]$ and V be given by (2.1). V can be considered as a velocity field which generates the Lagrangian coordinate (y,t) through

$$\begin{cases} x = x(y,t) \\ x_t = V(x,t) \end{cases}. \tag{2.5}$$

By (2.4) the curves $x = \zeta_i(t), i = 1,2$ become vertical so that they correspond to $y =$ constant in the new coordinate system. It turns out that (2.5) is satisfied if we define y by

$$y = \int_{\zeta_1(t)}^{x} u(s,t)\,ds. \tag{2.6}$$

Note that $y = 0$ corresponds to $x = \zeta_1(t)$ and $y = 1$ corresponds to $x = \zeta_2(t)$.

LEMMA 3. *Let y be defined by (2.6). The function $U(y,t) = u(x,t)$ satisfies*

$$U_t \geq -\frac{C}{t}U \qquad \text{in } (0,1)\times(0,T).$$

Next we define $v(x,t) = \frac{m}{m-1}u^{m-1}(x,t)$ and

$$\xi_i(t) = \lim_{x\to\zeta_i(t)} v_x(x,t). \tag{2.7}$$

We are now in a position to prove the main result of this section.

THEOREM 2. *For all $T > 0$, we have that*

$$(-1)^i\left\{\xi_i' + \frac{Cm}{t}\xi_i\right\} \leq 0, \qquad i = 1,2 \tag{2.8}$$

in $\mathcal{D}'(0,T)$, where the constant $C = C(T)$ has been defined in Lemma 2.

PROOF. We deduce from Lemma 3 that

$$(t^C U)_t \geq 0 \qquad \text{in } (0,1)\times(0,T). \tag{2.9}$$

Let $\psi \in C_0^\infty(0,T)$ be an arbitrary and nonnegative test function. We define $H \in C([0,1]) \cap C^1((0,1))$ by

$$H(y) = \int_0^T \psi'(t)t^{Cm}(U(y,t))^m\,dt.$$

By (2.9)

$$H(y) = -\int_0^T \psi(t)(t^{Cm}(U(y,t))^m)_t\,dt \leq 0.$$

Since on the other hand $H(0) = 0$, one can show that $D^+H(0)$ exists and is such that $D^+H(0) \leq 0$ which in turn implies that

$$(t^{Cm}\xi_1)_t \geq 0 \text{ in } \mathcal{D}'(0,T). \tag{2.10}$$

(2.10) implies (2.8) with $i = 1$.

THEOREM 3. *The one-sided derivatives $D^\pm\zeta_i(t)$ exist for every $t > 0$ and*

$$(-1)^i(\zeta_i'' + \frac{Cm}{t}\zeta_i') \geq 0, \qquad i = 1,2 \tag{2.11}$$

holds in $\mathcal{D}'(0,T)$.

PROOF. For convenience we restrict ourselves to the case $i = 1$. The inequality (2.10) implies that $t^{Cm}\xi_1$ is nondecreasing in $(0,T)$. We define

$$F(t) = \int_1^t s^{Cm}\zeta_1'(s)\,ds. \tag{2.12}$$

Since $\zeta_1'(t) = -\xi_1(t)$ for a.e. $t \in (0,T)$ we conclude that F is concave in $(0,T)$. Consequently $D^\pm F(t)$ exists for every $t > 0$ and $D^-F(t) \geq D^+F(t)$. Writing (2.12) in the form

$$F(t) = t^{Cm}\zeta_1(t) - \zeta_1(1) - Cm\int_1^t s^{Cm-1}\zeta_1(s)\,ds,$$

we conclude that $D^\pm\zeta_1(t)$ exists for every $t > 0$.

COROLLARY 2. *There exist waiting times* t_i^*, $i = 1, 2$ *such that* $\zeta_i(t) = \zeta_{0i}$ *on* $[0, t_i^*]$, $i = 1, 2$ *and* $(-1)^i \zeta_i(t)$, $i = 1, 2$ *is strictly increasing for* $t > t_i^*$.

PROOF. We have that

$$D^{\pm} F(t) = t^{Cm} D^{\pm} \zeta_1(t), \qquad \text{for every } t \in (0, T),$$

so that

$$D^- \zeta_1(t) \geq D^+ \zeta_1(t),$$

and also

$$D^{\pm} \zeta_1(t) \leq \left(\frac{\tau}{t}\right)^{Cm} D^{\pm} \zeta_1(\tau), \qquad \text{for } 0 < \tau < t.$$

This completes the proof of Corollary 2.

3. DIFFERENTIABILITY OF THE FREE BOUNDARY

The inequalities in (2.11) are the main ingredients for the proof of the differentiability of the interfaces after the waiting times. The results below are given for the left interface $\zeta = \zeta_1$, since the results for the right interface are completely similar.

THEOREM 4. *The interface* $x = \zeta(t)$ *is a* C^1 *function in the interval* (t^*, ∞), v *is a* C^∞ *smooth function in a neighborhood of* $x = \zeta(t)$, $t > t^*$ *restricted to* $x > \zeta(t)$ *and*

$$v(x,t) = v_\gamma(x - \zeta(t_0), t - t_0) + o(|x - \zeta(t_0)| + |t - t_0|),$$

where $t_0 > t^*$, $\gamma = \zeta'(t_0)$ *and* $v_\gamma(x,t) = \left[-\gamma x + \gamma^2 t\right]_+$. *Moreover* v_x *and* v_t *have limits as* $(x,t) \to (x_0, t_0)$ *with* $x > \zeta(t)$ *and* $x_0 = \zeta(t_0)$. *In particular we have that* $\zeta'(t) = -v_x$ *and* $v_t = v_x^2$ *on the set* $\{x = \zeta(t), t > t^*\}$.

In order to show the C^1 regularity of ζ for $t > t^*$, one proves that

$$D^- \zeta(t) = D^+ \zeta(t) \text{ for } t > t^*.$$

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